Notes on the simulation of the response of a photomultiplier tube

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Abstract

The simulation of the response of a photomultiplier tube (PMT) can be performed with a simple procedure, despite of its complexity. One just has to have very clear the nature of the whole process, step by step. In this small notes I try to clarify the physical processes involved.

Introduction

The simulation of the response of a photomultiplier tube (PMT) can be performed with a simple procedure, despite of the complexity of the process. These notes are written down simply to clarify the effects that have to be taken into account, like the Quantum Efficiency (QE) of the PMT (usually a function of the wavelength of the incident photon γ), the natural fluctuations of this QE for a given fixed wavelength, the first dynode response, the possible afterpulsing, the single photoelectron response of the PMT, etc. In this small note I will try to clarify some concepts about the simulation of the conversion of normal photons (in our case of Cherenkov astronomy, Cherenkov photons and photons coming from the Light of Night Sky and starlight) into number of photoelectrons after the photocathode.

1 Fluctuations taking place

Before we try to understand this process of conversion of Cherenkov photons into photoelectrons, let's try to identify the possible sources of fluctuations.

In the first place, we have a given number of incoming Cherenkov photons, which fluctuates. This is due to the statistical nature of the generation of the atmospheric shower. This means that even for a fixed energy of the primary particle which generated the shower, the probabilistic generation of secondary particles, the fluctuations in the height of the first interaction (and hence in the height where the maximum particle generation is achieved) and the random generation of Cherenkov photons by charged particles will lead to a fluctuating number of Cherenkov photons.

At the end of the cascading process, and after we introduce the simulation any given set of reflecting or guiding devices in our instrument, we will have an input number of photons in the photomultiplier tube. I will call this number of photons $\mathcal{N}_\gamma$.

Now we want to simulate a measurement of $\mathcal{N}_\gamma$ leading to a certain current I, or a given amount of electrons in the anode N_e. The Cherenkov photons hitting the photocathode will produce a given number of photoelectrons. This process is probabilistic, and depends on the QE of the photocathode. But this QE not only depends of the wavelength of the incident photon. It also depends on the specific place of the photocathode where the photon hits. Additionally, the measured QE of a PMT is just an average value over many photons. This means that the QE itself must be seen as a fluctuating term, in the sense of its probabilistic nature.

After the photoelectrons are emitted from the photocathode, a certain number of them will arrive at the first dynode. Of course, this depends on the place where the photoelectron come from, and on the design of the PMT itself. This is the so called *first dynode collection efficiency*. Different measurements lead to a value of about 90%. After hitting the first dynode, each photoelectron can liberate a typical number of 6 electrons, but again this can fluctuate.

We just started the cascading process. After the first dynode comes the second, where we again have a multiplicating term and an efficiency term. This multiplicating process continues until we reach the anode, where we finally have a big number of electrons, depending on the gain and the voltage of our PMT.

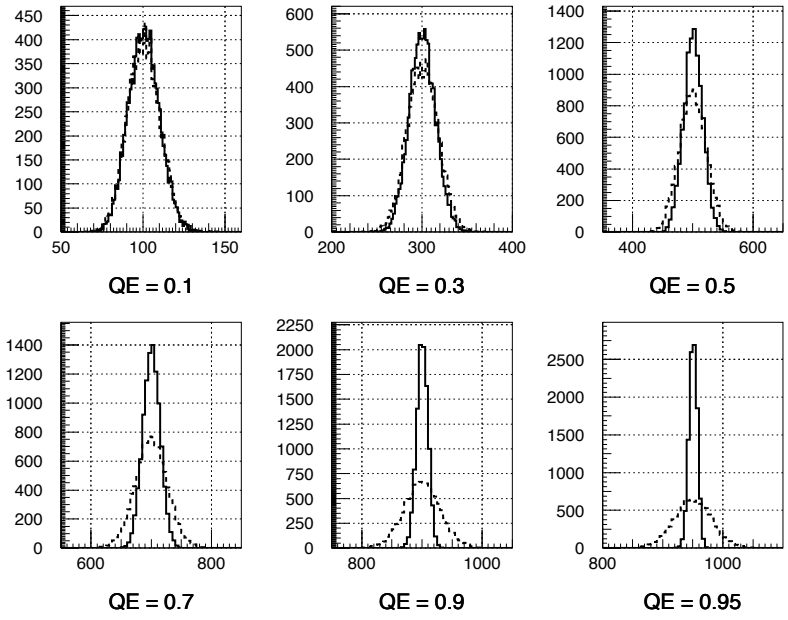

Figure 1: Distributions of the number of outcoming photoelectrons ($\mathcal{N}_{\rm ph.e.s}$, solid line; $\mathcal{N}^{\rm r}_{\rm ph.e.s}$, dashed line) for different values of $\mathcal{QE}$. In this case, $\mathcal{N}_\gamma = 1000$.

Al this cascade process can be simulated by using the so called *single photoelectron response* or *single electron spectrum* (SES) of the photomultiplier, which is nothing but the distribution of the output we get from the photomultiplier for a single photoelectron release by the photocathode. This depends also on the high voltage applied to the PMT. Therefore, by using this distribution, we can simulate the output of each single photoelectron, superpose all these output, and we will get at the end a realistic view of the response of our PMT to the incident amount of light.

2 Simulation of the Quantum Efficiency

In this document we assume that *a single photon cannot produce, for whatever processes inside the photocathode, more than one photoelectron.* For each single photon the generation of a photoelectron is therefore what is called a *Bernoulli* process: we can have only two outcomes: success (with probability p), or fail (with probability $1-p$). For a given number of incoming photons the generation of photoelectrons is a good example of a *binomial* process. Let's first forget about the dependency of the Quantum Efficiency with the wavelength. Let's assume, therefore, that we have a monochromatic bunch of photons. We call then $\mathcal{QE}$ =QE(λ_0), the Quantum Efficiency (the *average* value, obtained by measurements) of the photocathode at one fixed wavelength λ_0.

We see that each incoming photon have a certain probability $\mathcal{QE}$ of generating a photoelectron. This can be simulated by using a *uniformly distributed random number* X in the interval [0,1] for each photon: whenever this number X is smaller than $\mathcal{QE}$, we assume the photon generated a photoelectron; otherwise, it didn't. A simple algorithm to do this can is:

A1. Take photon.

A2. Generate uniform random number r in $(0,1)$.

A3. If $r < \mathcal{QE}$, take photon, else go to A1.

A4. Are there photons left? If yes, go to A1; else Stop.

Of course, after following this algorithm $\mathcal{N}_\gamma$ times we will have, *on average*, a number of photoelectrons:

$$\bar{\mathcal{N}}_{\rm ph.e.s} = \mathcal{N}_\gamma \times \mathcal{QE} \tag{1}$$

The words *on average* means simply that we will not get, from our experiment, exactly a number $\bar{\mathcal{N}}_{\rm ph.e.s}$ of photoelectrons. At least, not always. If we repeat this experiment a lot of times we will get instead a distribution of numbers $\mathcal{N}_{\rm ph.e.s}$, corresponding to the number of photoelectrons produced. This distribution, given the *binomial* nature of the process, will result in a *binomial distribution*. The mathematical expresion for this is:

$$\begin{aligned} \mathcal{P}_{\rm Binom}(r) &= \binom{N}{r} p^r q^{N-r} \\ &= \frac{N!}{r!(N-r)!} p^r (1-p)^{N-r} \end{aligned} \tag{2}$$

where $\mathcal{P}_{\rm Binom}(r)$ is the probability of getting r successes out of N independent trial, each of which has only two possible outcomes: success (with probability p) or failure (with probability $q = 1 - p$). We can show that the expectation value of the number of successes is $\bar{r} = \sum r\mathcal{P}(r) = Np$, which is far for surprising. For this distribution we have also that $\sigma^2 \leq \bar{r}$, where the equality holds only for p=0. In general the variance is smaller than the mean. This is so because the upper limit imposed on r (which cannot be larger than N) reduces the spread of the r-distribution.

The main use of the binomial distribution is in the limits:

1. $p \to 0$, $N \to \infty$, but $Np = \mu$ (constant), when Binomial $\longrightarrow$ Poisson
2. $p = const.$, $N \to \infty$, when Binomial $\longrightarrow$ Gaussian

In our case, $N = \mathcal{N}_\gamma$, $p = \mathcal{QE}$ and $Np = \bar{\mathcal{N}}_{\rm ph.e.s}$. This means that, *for $\mathcal{N}_\gamma$ large and in the case of $\mathcal{QE} \to 0$* (more generally speaking, for $\mathcal{QE}$ smal) the distribution of the number of outcoming photoelectrons $\mathcal{N}_{\rm ph.e.s}$ will be very similar to the distribution obtained by using the mean number of photoelectrons $\bar{\mathcal{N}}_{\rm ph.e.s}$ as the mean value of a Poisson distribution. More clearly, the requierements of $\mathcal{N}_\gamma$ large and $\mathcal{QE}$ small are needed if we want to use this procedure, what I will call the *Poisson approach*.

The use of the *Poisson approach* is very attractive because you don't need to work with separate photons. Moreover, some times its the most direct approach: you can get your photons at the entrance of the PMT in bunches of several tens of photons. In this case it's more comfortable to use mean values and the corresponding Poisson distribution. We are sure that N is going to be large enough. But we still have to be sure that our $\mathcal{QE}$ is small enough to allow us to use this approach. Fortunately, this is normally the case.

In order to show the possible deviations we can get by using the *Poisson approach*, A simple simulation was done, where a pre-fixed number of photons $\mathcal{N}_\gamma$ was sent to a hypothetic photocathode, where we simulated the Bernoulli process of the production of a photoelectron by using the simple algorithm before. The $\mathcal{QE}$ of the photocathode was also fixed, and we counted the number of photoelectrons emerging from it $\mathcal{N}_{\rm ph.e.s}$. In addition, using the number of incoming photons and $\mathcal{QE}$, we estimated the average number of photoelectrons $\bar{\mathcal{N}}_{\rm ph.e.s}$ we should get, and then followed the *Poisson approach* to get the final number of photoelectrons $\mathcal{N}^{\rm r}_{\rm ph.e.s}$.

In order to show how we can get severe deviations from the right behavior by using the *Poisson approach*, I performed the simulation varying the number of incoming photons and the value of $\mathcal{QE}$. The results are shown in Fig. 1, 3 2. In Fig. 1 we see the effect of using a $\mathcal{QE}$ very different

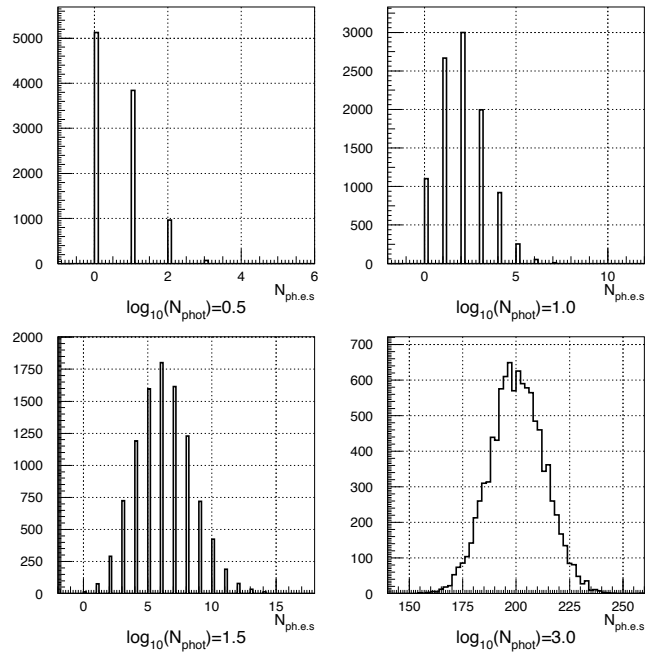

Figure 2: Distributions of the number of outcoming photoelectrons ($\mathcal{N}_{\rm ph.e.s}$) for different values of $\mathcal{N}_\gamma$ (in these cases $\log_{10}\mathcal{N}_\gamma = 0.5, 1.0, 1.5$ and 3.0), using a $\mathcal{QE} = 0.2$.

from 0. For small values of $\mathcal{QE}$, both distributions of photoelectrons $\mathcal{N}_{\rm ph.e.s}$ (solid line) and $\mathcal{N}^{\rm r}_{\rm ph.e.s}$ (Dashed line) appear reasonably similar. However, as $\mathcal{QE}$ is diverging from our hypothesis of $\mathcal{QE}$ small, both distributions get more and more different. In Fig. 3 we can see how different these distributions are for different $\mathcal{QE}$, by using the quantity

$$z = \frac{\sigma(\mathcal{N}^{\rm r}_{\rm ph.e.s})}{\sigma(\mathcal{N}_{\rm ph.e.s})} \tag{3}$$

Although the mean value remains practically the same for both distributions, no matter the value of $\mathcal{QE}$, the variance $\sigma^2(\mathcal{N}^{\rm r}_{\rm ph.e.s})$ increases much more than the variance $\sigma^2(\mathcal{N}_{\rm ph.e.s})$ with $\mathcal{QE}$ increasing. As we said before, this is just the result of the upper limit imposed in $\mathcal{N}_{\rm ph.e.s}$ (it cannot be larger than $\mathcal{N}_\gamma$).

3 Simulation of Single Electron Spectrum

This is just an application of the general procedure of getting a series of random numbers following a user-defined distribution, called the *Acceptance-Rejection Method* (Von Neumann). Let's assume we know the *single electron spectrum*, $S(x)$, defined in an interval (a, b). This will be in principle a not normalized function proportional to the normalized probability density function (p.d.f.) of distribution we want to simulate. Then, choose a p.d.f. uniform on the interval (a, b). Find a constant C such that C times this uniform p.d.f. is everywhere $\geq S(x)$. The situation is shown in Fig. 4.

Then, first, simulate an x uniformly on (a, b). Then generate a y on $(0, C/(b-a))$. The point (x, y) will uniformly populate the box shown. If $y \leq S(x)$, we accept x as then next value f

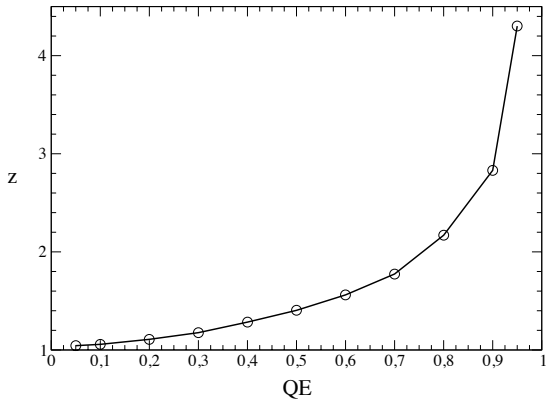

Figure 3: Variation with different $\mathcal{QE}$ of the quotient $z = \sigma(\mathcal{N}^{\rm r}_{\rm ph.e.s})/\sigma(\mathcal{N}_{\rm ph.e.s})$, also in the case of $\mathcal{N}_\gamma = 1000$. For another values of $\mathcal{N}_\gamma$ the curve is very similar.

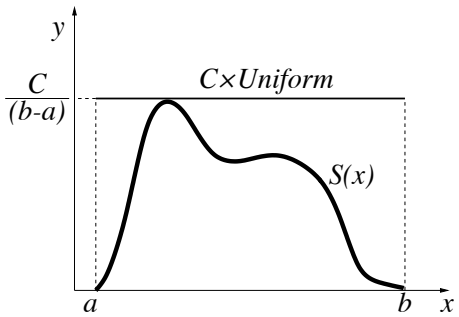

Figure 4: Ilustration of the *Acceptance-Rejection Method* (Von Neumann). The pairs of random numbers (x, y) will cover the area below the line $y = C/(b-a)$. Only those below the curve $S(x)$ will be accepted.

the random number. If $y \geq S(x)$, reject x and try again. This method is very simple and has an efficiency (fraction of values x accepted) of $\epsilon =$ (area under $S(x)$)/(area under bounds). (Note that if the function $S(x)$ has sharp peaks, the efficiency can be very low; one can then use different constants for different regions in the interval (a, b).)

For each photoelectron we will get, by using this procedure, the amplitude of the registered signal. In order to simulate a realistic signal at the output of our PMT, we should use the arrival time of each photon. On top of this time, the PMT introduces a delay. This delay follows a distribution similar to a gaussian. The time between the arrival of a delta-function light pulse and the time where the output signal reaches its maximum is called *electron transit time* (ETT). The delay between our input light and the output signal comes then governed by the ETT and the *transit time spread* (TTS, also called *transit time jitter*, the FWHM of the distribution of delays). The output signal is characterized by its *rise time* (time where the output signal rises from 10% to 90% of the maximum amplitude). Sometimes, instead of having the *rise time*, $t_{\rm rise}$ we have the FWHM of the output characteristic, $\approx 2.36\sigma$. The situation is schematized in Fig. 5. With all this we can reproduce the response of our PMT to the to the bunch of incident photons.

After all this chain of events, we can then introduce either the trigger logic of our system, or any electronic device which modifies this signal.

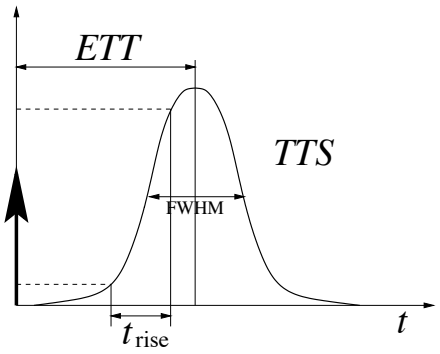

Figure 5: Ilustration of the output signal for an incoming delta-function light pulse. The main parameters of the time response of a PMT are shown.

References

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